A cluster-separable Born approximation for the 3D reduction of the three-fermion Bethe-Salpeter equation.

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July 16, 1999

Abstract

We perform a 3D reduction of the two-fermion Bethe-Salpeter equation based on Sazdjian's explicitly covariant propagator, combined with a covariant substitute of the projector on the positive-energy free states. We use this combination in the two fermions in an external potential and in the three-fermion problems. The covariance of the two-fermion propagators insures the covariance of the two-body equations obtained by switching off the external potential, or by switching off all interactions between any pair of two fermions and the third one, even if the series giving the 3D potential is limited to the Born term or more generally truncated. The covariant substitute of the positive energy projector preserves the equations against continuum dissolution without breaking the covariance.

PACS 11.10.Qr Relativistic wave equations.
PACS 11.10.St Bound and unstable states; Bethe-Salpeter equations.
PACS 12.20.Ds Specific calculations and limits of quantum electrodynamics.

Keywords: Bethe-Salpeter equations. Salpeter's equation. Breit's equation. Relativistic bound states. Relativistic wave equations.

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Contents

1 Introduction

The Bethe-Salpeter equation [1, 2] is the usual tool for the study of relativistic bound states. The principal difficulty in the treatment of this equation comes from the existence of unphysical relative time variables. Although the Bethe-Salpeter equation can be directly studied when the Bethe-Salpeter kernel is simple (one can perform the "Wick rotation") or replaced by a separable kernel, the most usual practice in the two-fermion problem consists in eliminating the relative time variable (3D reduction). This 3D reduction can be based on an instantaneous replacement of the Bethe-Salpeter kernel (by an expression independent of the relative energy), on an instantaneous replacement of the free Green function (by an expression combining a delta fixing the relative energy and a 3D propagator) or on both. The exact equivalence (in what concerns the physically measurable quantities of the pure two-fermion problem) with the original Bethe-Salpeter equation can be obtained by recuperating the difference with the original Bethe-Salpeter kernel and/or free Green function in a series of correction terms to the 3D potential.

The 3D reduction of the two-fermion Bethe-Salpeter equation has been performed by many authors [3-19]. All methods are theoretically equivalent at the limit of all correction terms included. Beyond the two-fermion problem, we have the cases of two fermions in an external potential ($2\frac{1}{2}-$body problem) and of three fermions (3-body problem). In these cases we meet new difficulties which

do not appear or can be easily solved in the two-fermion case: the Lorentz invariance / cluster separability problem, the continuum dissolution problem and the hermiticity problem.

— Lorentz invariance and cluster separability. It is always possible to render an equation Lorentz invariant by working in the general rest frame (center of mass reference frame) and by building invariants with the total 4-momentum vector, although the result may be unelegant and artificial. The Lorentz invariance requirement becomes a tool when combined with the cluster separability requirement: when all mutual interactions are "switched off", we must get a set of three free Dirac equations. This total separability can easily be obtained by using as hamiltonian the sum of three free Dirac hamiltonians and three mutual interaction terms. The real difficulty appears when only the interactions with fermion 3 (for example) are switched off. If we want a full cluster separability, the resulting equation for the (12) cluster can not refer to the global center of mass frame anymore, as the momentum of fermion 3 enters in the definition of this frame.

Lorentz invariance and cluster separability can be explicit or implicit (via rearrangements). The best known example of implicit Lorentz invariance is a free Dirac equation solved with respect to the energy: it becomes explicitly covariant by multiplication with the β matrix. Other implicit Lorentz invariances are not that trivial. For example, the 3D reductions of a Bethe-Salpeter equation are implicitly covariant, provided the series generated by this reduction are not truncated.

In fact, the homogeneous Bethe-Salpeter equation from which we shall start does not obey cluster separability: this equation is valid for two-fermion bound states only, i.e. for a total energy below the continuum. Our 3D equation will thus be used in the computation of three-fermion bound states. A priori, it will not necessarily give also the correct scattering amplitudes or obey cluster separability, even with all higher-order terms included. We shall nevertheless try to get a 3D equation which is also valid in the contunuum region.

— Solution of the continuum dissolution problem. In the relativistic equations for several relativistic particles, the physical bound states are degenerate with a continuum of states combining asymptotically free particles with opposite energy signs. This often neglected fact forbids the building of normalizable solutions in the $N > 2$-body problem (including the two-body plus potential problem. In the pure two-body problem the mixing is prevented by the conservation of the total momentum). The usual solution consists in including positive-energy projection operators into the zero-order propagator [20-26]. The modified equations must of course continue to satisfy the other requirements, like the Lorentz invariance / cluster separability requirement.

— Hermiticity and total energy independence of the interaction terms. In the two-body problem, nonhermitian interaction terms can be hermitian with respect to a modified scalar product, or be made hermitian via a rearrangement of the equation. In the three-body problem these rearrangements could be more

complicated. The hermiticity and the independence on the total energy are linked features, as one of these is often achieved at the expense of the other one. Energy depending interaction terms destroy some of the advantages of the use of an hermitian hamiltonian, such as the mutual orthogonality of the solutions, and leads to modify the usual scalar products and perturbation calculation methods. The 3D potentials deduced from field theory are generally energy dependent, at least in the higher-order terms. We shall require hermiticity and energy-independence (or slow energy dependence) below the inelastic threshold in the lowest order terms at least.

In our preceding works on the $2\frac{1}{2}-$body problem [23], and. more recently, on the three-fermion problem [26], our approach was based on the cluster separability requirement. In the two fermions in an external potential problem, the free Green function was written in terms of the Dirac free hamiltonians, to which the external potential was added for each fermion (in the laboratory reference frame). The 3D reduction was then performed exactly as in the pure two-fermion case. In the three-fermion problem the 3D potential was obtained by adding the three 3D potentials obtained by the 3D reduction of the three two-fermion equations in the three-fermion rest frame. Each two-fermion potential depending in general of the total energy of the subsystem, this two-fermion energy was taken as the three-fermion total energy, minus the free Dirac hamiltonian of the spectator fermion. The continuum dissolution was avoided by choosing a two-fermion reduction based on a 3D propagator containing a projector on the positive eigenvalues eigenstates of the Dirac's free hamiltonian (modified in the $2\frac{1}{2}-$body problem by including the external potential). With these choices the three two-cluster limits are exact: switching off the external potential or switching off the two interaction terms with a spectator fermion gives the exact 3D equation which would be obtained by the reduction of the two-body Bethe-Salpeter equation (and switching off the mutual interaction in the $2\frac{1}{2}-$body problem gives two independent equations describing each a single fermion in a potential). From this exact 3D equation for two fermions it is possible to go back to the original two-fermion Bethe-Salpeter equation and to perform the 3D reduction again in another reference frame, so that we can consider this two-fermion equation as implicitly covariant. This covariance becomes an approximated covariance when the series giving the 3D potential is truncated, for example by keeping only the Born term.

Although our 3D equations have exact two-cluster limits, they are themselves approximations. In the $2\frac{1}{2}-$body problem, one should take into account the modifications brought by the external potential to the fermion propagators also inside the Bethe-Salpeter irreducible kernel. In the three-body problem one has to take into account the irreducible three-body terms at the Bethe-Salpeter equation level and also the three-body terms generated at the 3D level by the reduction. We tried to do that recently [27].

In the present work we try to get explicitly covariant two-fermion cluster limits by using Sazdjian's explicitly covariant propagator. In this way, each

term of the series giving the 3D potential is explicitly covariant and an approximation of this potential (such as Born's approximation) is possible without breaking the covariance. There is a problem with the fact that Sazdjian's equation in its original form has an interaction term which is not hermitian. We have to transform Sazdjian's two-fermion equation into an equivalent equation with an hermitian interaction term before combining the three two-fermion interaction terms into a three-fermion 3D equation. Another problem comes from the projection operator on the positive eigenvalues eigenstates of the free Dirac hamiltonians (introduced in order to avoid continuum dissolution beyond the two-fermion problem): this projector is not covariant so that it is necessary to search for a covariant substitute of it.

In section 2 we introduce the notations and a standard reduction method of the two-fermion Bethe-Salpeter equation. In section 3 we present Sazdjian's covariant propagator, transform Sazdjian's equation into an equivalent equation with an hermitian interaction term, and build a covariant substitute of the positive-energy projector. A subsection is devoted to the one-body limit of the resulting two-fermion equation. This explicitly covariant two-fermion equation is exploited in sections 4 ($2\frac{1}{2}-$body problem) and 5 (three-body problem). Section 6 is devoted to conclusions.

2 Two fermions.

2.1 Notations.

We shall write the Bethe-Salpeter equation for the bound states of two fermions [1] as

$$\Phi = G^0 K \Phi, \tag{1}$$

where Φ is the Bethe-Salpeter amplitude, function of the positions x_1, x_2 or of the momenta p_1, p_2 of the fermions, according to the representation chosen. The operator K is the Bethe-Salpeter kernel, given in a non-local momentum representation by the sum of the irreducible two-fermion Feynman graphs. The operator G^0 is the free propagator, given by the product $G_1^0 G_2^0$ of the two individual fermion propagators

$$G_i^0 = \frac{1}{p_{i0} - h_i + i\epsilon h_i}\,\beta_i = \frac{p_{i0} + h_i}{p_i^2 - m_i^2 + i\epsilon}\,\beta_i \tag{2}$$

where the h_i are the Dirac free hamiltonians

$$h_i = \vec{\alpha}_i\,.\vec{p}_i + \beta_i\, m_i \qquad (i = 1, 2). \tag{3}$$

We shall define the total (or external, CM, global) and relative (or internal) variables:

$$X = \frac{1}{2}(x_1 + x_2)\ , \qquad P = p_1 + p_2\ , \tag{4}$$

$$x = x_1 - x_2 \ , \qquad p = \frac{1}{2}(p_1 - p_2). \tag{5}$$

and give a name to the corresponding combinations of the free hamiltonians:

$$S = h_1 + h_2 \ , \quad s = \frac{1}{2}(h_1 - h_2). \tag{6}$$

We know that, at the no-interaction limit, we shall have to get a pair of free Dirac equations:

$$(p_{10} - h_1)\Psi = 0, \qquad (p_{20} - h_2)\Psi = 0, \tag{7}$$

where Ψ depends on x_1, x_2. Let us also write their iterated versions

$$(p_{10}^2 - E_1^2)\Psi = 0, \qquad (p_{20}^2 - E_2^2)\Psi = 0 \tag{8}$$

with

$$E_i = \sqrt{h_i^2} = (\vec{p}_i^{\,2} + m_i^2)^{\frac{1}{2}}. \tag{9}$$

Interesting combinations can be obtained from the sum and differences of the equations (7) or of the iterated equations (8):

$$(P_0 - S)\Psi = 0, \qquad (p_0 - s)\Psi = 0, \tag{10}$$

$$H^0\Psi = 0, \qquad (p_0 - \mu)\Psi = 0 \tag{11}$$

with

$$H^0 = 2[(p_1^2 - m_1^2) + (p_2^2 - m_2^2)]_{p_0=\mu} \ = \ P_0^2 - 2(E_1^2 + E_2^2) + 4\mu^2, \tag{12}$$

$$\mu = \frac{1}{2P_0}(E_1^2 - E_2^2) = \frac{1}{2P_0}(h_1^2 - h_2^2) = \frac{sS}{P_0}. \tag{13}$$

2.2 3D reduction of the two-fermion Bethe-Salpeter equation.

In the zero-order approximation, the free propagator G^0 will be replaced by a carefully chosen "propagator" G^δ, combining a constraint like $\delta(p_0 - \mu)$ fixing the relative energy, and a global 3D propagator like $-2i\pi(P_0 - S)^{-1}\beta_1\beta_2$. The argument of the δ and the inverse of the 3D propagator should be combinations of the operators used in the free equations (at last approximately and for the positive-energy solutions). There exists an infinity of possible combinations [3-18]. The best choice depends on the quantities one wants to compute (energy of the lowest state, hyperfine splitting, recoil of a nucleus, etc...) and on the properties one wants to preserve exactly in the first approximation (cluster separability, Lorentz invariance, heavy mass limits, charge conjugation symmetry...). All choices would be equivalent if all correction terms could be computed, but this is of course impossible.

We shall see below that the reduced wave function Ψ (from which the relative time-energy degree of freedom can be trivially eliminated) is given by

$$\Psi = G^{\delta}(G^0)^{-1}\Phi. \tag{14}$$

The first choice to be made in the 4D $\rightarrow$ 3D reduction is that of the constraint which fixes the 3D hypersurface ($p_0 = \mu$ for example) on which we want to work. The remaining of G^{δ} is a purely 3D operator, the different choices of which result in different 3D operators applied on a common basic 3D wave function and in different rearrangements of a common reduction series giving the 3D potential. The various propagator-based 3D reductions of the literature can thus be classified according to the constraint they use. Once this constraint chosen, we can only write different equivalent forms (or sometimes projections) of the same 3D equation. It is the unavoidable truncation of the reduction series which makes the difference (numeric, if we simply want to compute predictions of field theory; more fundamental, if we want to write constraint theory equations with QCD inspired potentials).

Two natural choices for the constraint are $\delta(p_0 - s)$, based on the first-order equations (10) and $\delta(p_0 - \mu)$, based on the second-order equations (11). In $\delta(p_0 - s)$, s is an operator which can be diagonalized in momentum space using the four projectors on the subspaces corresponding to the different signs of h_1, h_2 (see below). By contrast, μ is a momentum-depending number, which reduces to $(m_1^2 - m_2^2)/2P_0$ in the two-fermion center of mass frame $\vec{P} = 0$, conserved in a pure two-fermion problem (but not in the two-fermion plus potential problem or in the three-fermion problem). Other constraints could also be chosen, such as that of Gross [6], which puts one particle (normally the heaviest) on mass shell. We made a nonexhaustive review in ref.[17].

If we consider the contributions of the poles of

$$G^0 = \frac{1}{\frac{1}{2}P_0 + p_0 - h_1 + i\epsilon h_1} \, \frac{1}{\frac{1}{2}P_0 - p_0 - h_2 + i\epsilon h_2} \beta_1 \beta_2 \tag{15}$$

in an expression like KG^0K, we must perform an integration with respect to p_0. If K is instantaneous, we get

$$G^0 \, \rightarrow \, G^S \, = \, \int dp_0 G^0(p_0) \, = \, \tau G^B,$$

$$G^B \, = \, -2i\pi \, g^0 \, \beta_1 \beta_2, \qquad g^0 = \, \frac{1}{P_0 - S + i\epsilon P_0} \tag{16}$$

with

$$\tau = \frac{1}{2}(\tau_1 + \tau_2), \qquad \tau_i = \frac{h_i}{\sqrt{h_i^2}} = \frac{h_i}{E_i} = \mathrm{sign}(h_i). \tag{17}$$

or

$$\tau = \Lambda^{++} - \Lambda^{--}, \qquad \Lambda^{ij} = \Lambda_1^i \Lambda_2^j, \qquad \Lambda_i^{\pm} = \frac{E_i \pm h_i}{2E_i}. \tag{18}$$

When K is not instantaneous, we must add the contributions of its singularities. Furthermore, in the residues of the poles of G^0 we must take K at $p_{10}=h_1$ or at $p_{20}=h_2$, according to the chosen integration path and to the sign of τ. In order to work with a single G^δ, we shall share the "offmassshellness" equally between the two fermions. In [26], we fixed p_0 to s, which means $(p_{10}-h_1) = (p_{20}-h_2)$. Here, we shall fix p_0 to μ, which means $(p_{10}^2-h_1^2) = (p_{20}^2-h_2^2)$:

$$G^\delta(p_0) = \delta(p_0-\mu)\, G^S. \tag{19}$$

The operator τ has a clear meaning in the basis built with the free solutions: it is +1 for $h_1, h_2 > 0$, -1 for $h_1, h_2 < 0$ and zero when they have opposite signs. It comes from the dependence of the p_0 integral on the signs of the ϵh_i.

There exists an infinity of possible choices for the propagator G^δ. They must however be (at least approximately) identical on the positive energy mass shell $P_0 = E_1 + E_2$. An obvious simplification can be made by replacing τ by 1 (its value for the physical free solutions) or by $\epsilon(P_0)$ (to get the correct value for the corresponding antiparticle states too). The merits of the τ or no-τ choice are the matter of an old debate. The operator τ brings apparently useless complications in the pure two-body case. In the two-body plus potential problem, however, the generalization of this operator prevents the "continuum dissolution" disease (see below). We shall refer to G^S as Salpeter's 3D propagator [3] and to G^B as Breit's 3D propagator [28]. We could also keep only the Λ^{++} part of τ :

$$G^{++} \;=\; \Lambda^{++}\, G^B. \tag{20}$$

In the remaining of this subsection we shall work with a slightly more general zero-order propagator

$$G^\delta \;=\; \delta(p_0-\mu)\, A\, G^B \tag{21}$$

where A is an operator which commutes with g^0, such as τ (Salpeter), 1 (Breit) or Λ^{++}.

We shall write the free propagator as the sum of the zero-order propagator, plus a remainder:

$$G^0 = G^\delta + G^R. \tag{22}$$

The Bethe-Salpeter equation becomes then the inhomogeneous equation

$$\Phi = G^0 K \Phi = (G^\delta + G^R) K \Phi = \Psi + G^R K \Phi, \tag{23}$$

with

$$\Psi = G^\delta K \Phi \qquad (= G^\delta (G^0)^{-1} \Phi). \tag{24}$$

Solving (formally) the inhomogeneous equation (23) with respect to Φ and putting the result into (24), we get

$$\Psi = G^\delta K (1 - G^R K)^{-1} \Psi = G^\delta K^T \Psi \tag{25}$$

where

$$K^T = K(1 - G^R K)^{-1} = K + KG^R K + ... = (1 - KG^R)^{-1} K \tag{26}$$

obeys

$$K^T = K + KG^R K^T = K + K^T G^R K. \tag{27}$$

The reduction series (26) re-introduces in fact the reducible graphs into the Bethe-Salpeter kernel, but with G^0 replaced by G^R. Equation (25) is a 3D equivalent of the Bethe-Salpeter equation. It depends on the choice of G^δ.

The relative energy dependence of eq. (25) can be easily eliminated:

$$\Psi = \delta(p_0 - \mu)\, \psi \tag{28}$$

and ψ obeys:

$$\psi \;=\; \frac{A}{P_0 - S + i\epsilon P_0}\, V\, \psi \tag{29}$$

where V is proportional to K^T with the initial and final relative energies fixed to μ :

$$V \;= -2i\pi\, \beta_1 \beta_2 K^T(\mu, \mu). \tag{30}$$

In less compact but more precise notations:

$$\beta_1 \beta_2 K^T(\mu, \mu) \;\equiv\; \int dp_0' dp_0 \delta(p_0' - \mu) \beta_1 \beta_2 K^T(p_0', p_0) \delta(p_0 - \mu). \tag{31}$$

Note that we write (p_0', p_0) but (μ, μ), as we keep μ in operator form. This operator is diagonal in the spatial momentum space. The eigenvalue will depend on the position of μ in the formula: the eigenvalue of the first μ in (31) will be built with the final momenta and that of the last μ will be built with the initial momenta.

The inversion of the reduction is given by

$$\Phi \;=\; (\,1 - G^R K\,)^{-1}\, \Psi \;=\; (\,1 + G^R K^T\,)\, \Psi \;=\; (\,1 + G^0 K^T - G^{\,\delta} K^T\,)\, \Psi \;=\; G^0 K^T\, \Psi \tag{32}$$

or, explicitating the relative energy variable

$$\Phi(p_0') \;=\; G^0(p_0')\, K^T(p_0', \mu)\, \psi. \tag{33}$$

The splitting of G^0 into two terms containing a δ is the origin of unphysical singularities in the terms of K^T when the argument of the delta vanishes on the singularities of K. When the full K^T is computed, the singularities of the different terms cancel mutually. When K^T is truncated, some of the unphysical singularities have to be removed by hand [6, 24].

The propagator G^δ and thus G^R, K^T and V depend on the choice of A. We shall therefore add a superscript $\tau, B, ++$ when needed, for $A = \tau, 1, \Lambda^{++}$ respectively.

When $A = 1$ the operator $AV = V$ is hermitian for a fixed P below the inelastic threshold, with the usual scalar product. When $A = \Lambda^{++}$ the operator $\Lambda^{++}V^{++}$ is hermitian in the $\Lambda^{++}=1$ subspace, and we can write

$$\psi \,=\, \frac{1}{P_0 - E_1 - E_2 + i\epsilon}\, \Lambda^{++}\, V^{++}\, \Lambda^{++}\psi. \tag{34}$$

When $A = \tau$ the operator τV^{τ} is hermitian in the $\tau^2 = 1$ subspace with the scalar product

$$(\psi_i, \psi_j) \,=\, \int d^3p\; \psi_i^{+}(\vec{p})\, \tau(\vec{p})\, \psi_j(\vec{p}) \tag{35}$$

and we can write

$$\psi \,=\, \frac{1}{P_0 - S + i\epsilon P_0}\, \tau\, V^{\tau}\, \tau^2 \psi. \tag{36}$$

The 3D equations corresponding to a given constraint are related with each other and with the off mass shell transition matrix element at the 3D level. Starting from Breit's equation, we have

$$\psi \,=\, A\,\psi^B, \qquad V \,=\, V^B \left[1 - (1 - A)\, g^0 V^B\right]^{-1} \tag{37}$$

and, conversely

$$\psi^B \,=\, A^{-1}\,\psi, \qquad V^B \,=\, V\left[1 - (A - 1)\, g^0 V\right]^{-1} \tag{38}$$

if A^{-1} exists. If not, we can use

$$\psi^B \,=\, g^0\, V\, \psi \tag{39}$$

The 3D off mass shell transition matrix element $T^{3D} = -2i\pi\beta_1\beta_2 T(\mu, \mu)$ can be obtained by putting $A=0$ in (37):

$$T^{3D} \,=\, V^B \left[1 \,-\, g^0 V^B\right]^{-1} \tag{40}$$

and its Λ^{++} projection is

$$\Lambda^{++}\, T^{3D}\, \Lambda^{++} \,=\, \Lambda^{++}\, V^{++}\, \Lambda^{++} \,\left[1 \,-\, g^0\, \Lambda^{++}\, V^{++}\, \Lambda^{++}\right]^{-1}. \tag{41}$$

Until now we did not specify a reference frame, and our equations are not manifestly covariant under the Lorentz group. However:

– Trough not manifestly covariant, equations like (25) can always be transformed back into the original Bethe-Salpeter equation, provided the reduction series (26) is not truncated. The inclusion of higher-order terms will thus, in principle, improve an approached covariance.

– Even truncated, our equations could be made formally invariant by introducing a unit vector $n = (1, \vec{0})$ (we can call this vector the laboratory time unit) and by making all elements in our equations invariant by using scalar

products with n. If we want more than a formal invariance, this unit vector can however not be external to the system: one must therefore define n as $P/\sqrt{P^2}$ (the time unit of the center of mass reference frame), assuming $P^2 > 0$. An equation like (34) can thus be assumed to have been written in the center of mass reference frame and made explicitly covariant by using the $P/\sqrt{P^2}$ vector. This covariant form could be used to write the system of equations in other reference frames. There is no reason to do that in the pure two-body case, but it will become necessary when other objects (as an external potential or a third particle) are present. The $\delta(p_0 - \mu)\, G^B$ combination written in the center of mass reference frame looses its simplicity in its covariant form.

3 Towards covariant truncations.

3.1 Sadjian's explicitly covariant equation.

It would be interesting to be able to simplify the equations in a covariance-preserving way. With Sazdjian's explicitly covariant two-fermion propagator we shall present now it will become possible to truncate the reduction series in a covariant way (one might for example keep only the first term of the reduction series at the ladder approximation, i.e. the Born term). Writing G^0 in the form

$$G^0 = \frac{1}{(p_{10}^2 - E_1^2 + i\epsilon)(p_{20}^2 - E_2^2 + i\epsilon)}\,(p_{10} + h_1)(p_{20} + h_2)\beta_1\beta_2, \qquad (42)$$

forgetting for a while the matricial product $(p_{10} + h_1)(p_{20} + h_2)\beta_1\beta_2$ which is specific to the first-order equation and integrating the remainder with respect to p_0, we get a second-order equivalent of Salpeter's propagator:

$$G^{S2} \;=\; \frac{-4i\pi\sigma}{P_0 H^0} \qquad (43)$$

where

$$\sigma = \frac{1}{2}(\sigma_1 + \sigma_2), \qquad \sigma_1 = \frac{P_0 + 2\mu}{2E_1}, \qquad \sigma_2 = \frac{P_0 - 2\mu}{2E_2}. \qquad (44)$$

is the second-order equivalent of τ : for a given P_0, the operator σ_i is the off mass shell extrapolation of the sign of the energy of the fermion i. The presence of σ annihilates thus the residues of the poles at $P_0 = \pm(E_1 - E_2)$, so that G^{S2} can be written

$$G^{S2} \;=\; -2i\pi\,\frac{E_1 + E_2}{2E_1E_2}\;\frac{1}{P_0^2 - (E_1 + E_2)^2 + i\epsilon}\,. \qquad (45)$$

It is important to note the difference of action between τ and σ : the P_0-independent operator τ kills the subspaces of the $(+-)$ and $(-+)$ solutions (τ^2 is a projector), while the P_0-dependent operator σ simply annihilates the residues of

the corresponding poles in the free propagator. If we replace σ by $\epsilon(P_0)$ we get Sazdjian's second-order 3D propagator [12] :

$$G^{SZ2} \;=\; \frac{-4i\pi}{|P_0|\,H^0}. \tag{46}$$

The product of the second-order constraint $\delta(p_0-\mu)$ with Sazdjian's second-order 3D propagator (Sazdjian's second-order propagator) is invariant. If we now re-introduce the $(p_{10}+h_1)(p_{20}+h_2)\beta_1\beta_2$ factor, we get

$$\delta(p_0-\mu)\,G^{SZ} \;=\; -2i\pi\;\frac{(p_1\cdot\gamma_1+m_1)\,(p_2\cdot\gamma_2+m_2)}{p_1^2+p_2^2-(m_1^2+m_2^2)+i\epsilon}\;\delta\,(P\cdot p-\frac{m_1^2-m_2^2}{2}\,). \tag{47}$$

and Sazdjian's first-order 3D propagator [12] is thus

$$G^{SZ} \;=\; \frac{-i\pi}{|P_o|}\;\frac{2(p_{10}+h_1)(p_{20}+h_2)}{[\,(p_{10}^2-h_1^2)+(p_{20}^2-h_2^2)]}\;\beta_1\beta_2 \tag{48}$$

taken at $p_0=\mu,$ which means $(p_{10}^2-h_1^2)=(p_{20}^2-h_2^2)$. We have

$$\frac{2(p_{10}+h_1)(p_{20}+h_2)}{[\,(p_{10}^2-h_1^2)+(p_{20}^2-h_2^2)]} \;=\; \frac{p_{10}+h_1}{p_{20}-h_2} \;=\; \frac{p_{20}+h_2}{p_{10}-h_1} \;=\; \frac{P_0+S}{P_0-S} \tag{49}$$

where we took the sum of the numerators and denominators, so that

$$G^{SZ} \;=\; A^{SZ}\,G^B, \qquad A^{SZ} \;=\; \frac{P_0+S}{2\,|P_0|}. \tag{50}$$

Sazdjian's 3D equation is thus

$$\psi^{SZ} \;=\; \frac{1}{P_0-S+i\epsilon P_0}\;\frac{P_0+S}{2\,|P_0|}\,V^{SZ}\,\psi^{SZ}. \tag{51}$$

The equations built with Salpeter's or Breit's or with the positive-energy propagator were not explicitly covariant. They were however implicitly covariant, as it was always possible to go back to the original covariant Bethe-Salpeter equation and to perform the same 3D reduction again in another reference frame. However, this is possible only if the series giving K^T is not truncated. With a truncated K^T the squared mass spectrum $P^2=P_0^2-\vec{P}^2$ will not be independent of the spatial momentum $\vec{P}$. In contrast, with a 3D equation like (51), based on a covariant propagator, it becomes possible to change the reference frame without going back to the original Bethe-Salpeter equation. Even with a truncated V^{SZ}, the squared mass spectrum will be independent of the spatial momentum $\vec{P}$ and will be more easily computable by choosing the center of mass reference frame.

3.2 Restoring hermiticity.

In Sazdjian's 3D equation (51) the interaction term is not hermitian. We can transform this equation into an equation with an hermitian interaction by writing it in pseudo-Breit form with the definitions

$$\psi^{\overline{B}} \,=\, \frac{2\,P_0}{P_0+S}\,\psi^{SZ}, \tag{52}$$

$$V^{\overline{B}} = \epsilon(P_0)V^{SZ}\left[1-(\frac{P_0+S}{2P_0}-1)\frac{1}{P_0-S}\epsilon(P_0)V^{SZ}\right]^{-1} = \frac{2\,P_0V^{SZ}}{2\,|P_0|+V^{SZ}} \tag{53}$$

so that

$$\psi^{\overline{B}} \,=\, \frac{1}{P_0-S+i\epsilon P_0}\;\frac{2\,P_0V^{SZ}}{2\,|P_0|+V^{SZ}}\;\psi^{\overline{B}} \tag{54}$$

where we first moved an $\epsilon(P_0)$ from A^{SZ} to V^{SZ} in (51), whence the superscript $\overline{B}$ and the name "pseudo-Breit". Equation (54) can also be obtained more directly [29] by writing equation (51) in the form

$$P_0\left(1-\frac{V^{SZ}}{2\,|P_0|}\right)\psi^{SZ} \,=\, S\left(1+\frac{V^{SZ}}{2\,|P_0|}\right)\psi^{SZ} \tag{55}$$

and defining

$$\psi^{\overline{B}} \,=\, \left(1+\frac{V^{SZ}}{2\,|P_0|}\right)\psi^{SZ}. \tag{56}$$

While a truncation of the series giving directly V^B or $V^{\overline{B}}$ would spoil the covariance of the equation, a truncation of the series giving V^{SZ} would preserve this covariance. We could thus, for example, replace V^{SZ} by its Born approximation. A possible problem could arise from the zero's of the denominator of (53).

Another way of getting an hermitian interaction from (51) is provided by the transformation

$$\psi^{SZ} \,=\, \sqrt{\frac{P_0+S}{2\,|P_0|}}\,\chi \tag{57}$$

which leads to

$$\chi \,=\, \frac{1}{P_0-S+i\epsilon P_0}\sqrt{\frac{P_0+S}{2\,|P_0|}}\,V^{SZ}\sqrt{\frac{P_0+S}{2\,|P_0|}}\,\chi \tag{58}$$

but is possible only when $P_0+S>0$.

3.3 Covariant positive-energy projectors.

The combination of two fermions with opposite energy signs leads to continuum dissolution when an external potential or a third particle is added. This is the case with Breit's and Sazdjian's equations, while in other equations this difficulty is avoided owing to operators like τ, Λ^{++} or σ, which kill the unwanted singularities. Unfortunately, these operators are not covariant. They could not be introduced into Sazdjian's potential V^{SZ} without spoiling the invariance of the truncations.

A Lorentz invariant operator forcing a positive energy for fermion i could be

$$\theta_i \,=\, \theta(p_{i0})\theta(p_i^2) = \theta(p_{i0} - |\vec{p}_i|) \tag{59}$$

as the sign of the energy is invariant for positive squared masses. Taking the product of two such operators (one for each fermion) and replacing p_0 by μ gives

$$\hat{\theta} \,=\, \theta\left(\frac{P_0}{2} \,+\, \frac{\vec{p}_1^{\,2} - \vec{p}_2^{\,2}}{2P_0} \,-\, |\,\vec{p}_1| \,+\, \frac{m_1^2 - m_2^2}{2P_0}\right)\theta\left(\frac{P_0}{2} \,+\, \frac{\vec{p}_2^{\,2} - \vec{p}_1^{\,2}}{2P_0} \,-\, |\,\vec{p}_2| \,+\, \frac{m_2^2 - m_1^2}{2P_0}\right). \tag{60}$$

Let us assume that the heaviest fermion is fermion 2. We can then write (60) as

$$\hat{\theta} \,=\, \theta\left(\frac{P_0}{2} \,+\, \frac{\vec{p}_1^{\,2} - q_2^2}{2P_0} \,-\, |\,\vec{p}_1|\right)\theta\left(\frac{P_0}{2} \,+\, \frac{q_2^2 - \vec{p}_1^{\,2}}{2P_0} \,-\, |\,\vec{p}_2|\right) \tag{61}$$

with

$$q_2 \,=\, \sqrt{\vec{p}_2^{\,2} + m_2^2 - m_1^2}\,. \tag{62}$$

The sum of the arguments of the two θ is $P_0 - |\,\vec{p}_1| - |\,\vec{p}_2|$. The argument of the first θ is $(\,P_0 - |\,\vec{p}_1| + q_2)(\,P_0 - |\,\vec{p}_1| - q_2)/2P_0$, which implies that P_0 must be outside the interval $(|\,\vec{p}_1| - q_2, |\,\vec{p}_1| + q_2)$. The combination of these two results implies $P_0 > |\,\vec{p}_1| + q_2$. The second θ brings no supplementary restriction, its argument being always positive when $P_0 > |\,\vec{p}_1| + q_2$. We can thus write

$$\hat{\theta} \,=\, \theta\,(\,P_0 - |\,\vec{p}_1| - \sqrt{\vec{p}_2^{\,2} + m_2^2 - m_1^2}\,). \tag{63}$$

We see that our projection operator introduces a cutoff on the high $|\,\vec{p}_i|$ for a given P_0.

It is interesting to write $\hat{\theta}$ also in terms of the h_i, for comparison with the Λ^{++} projector and also for future use in the two-fermion plus potential problem. We get

$$\theta_i \,=\, \theta(p_{i0})\theta(p_i^2) \,=\, \theta(p_{i0})\,\theta(p_{i0}^2 - h_i^2 + m_i^2) \tag{64}$$

so that

$$\hat{\theta} \,=\, \left[\theta(p_{10})\,\theta(p_{10}^2 - h_1^2 + m_1^2)\,\theta(p_{20})\,\theta(p_{20}^2 - h_2^2 + m_2^2)\right]_{p_0=\mu}\,. \tag{65}$$

The constraint gives

$$p_{10} \,=\, \frac{P_0}{2} + \frac{h_1^2 - h_2^2}{2P_0}, \qquad p_{20} \,=\, \frac{P_0}{2} + \frac{h_2^2 - h_1^2}{2P_0} \tag{66}$$

and also $p_{10}^2 - h_1^2 \,=\, p_{20}^2 - h_2^2$ so that the last θ which concerns the heaviest fermion can be omitted. For a given bound state energy P_0, $\hat{\theta}$ eliminates the mixed-energy signs continuum which would share this energy. When $P_0 = S$, (65) reduces indeed to $\theta(h_1)\theta(h_2)$. In fig. 1, we draw a map of the $\hat{\theta}=1$ region in the (h_1, h_2) plane, for P_0 fixed to 2 (in arbitrary units) and different values of the lowest mass m_1 : 0, 0.5, 1 (solid line), 1.5, 2, 2.5 and 3. We draw only the first quadrant, the other ones being the mirror images of the first one. The $\hat{\theta}=1$ region is bounded by the two $p_{i0}=0$ curves, and by the $p_{10}^2 - h_1^2 + m_1^2 = 0$ curve corresponding to the lowest mass m_1. In each quadrant, this region is divided into four parts, corresponding to the signs of $h_1^2 - m_1^2$ and $h_2^2 - m_2^2$ (as an exemple, we indicate this partition in the $m_1 = 1, m_2 = 1.1$ case, for which P_0 is thus 0.1 below the threshold 2.1). Both expressions are positive definite for the pure two (and three) fermion systems, the negative values being for the bound states of the fermions in an external potential.

While the $\Lambda^{++} = 1$ region is simply the first quadrant, the $\hat{\theta} = 1$ region is divided into four symmetric parts, one for each quadrant, but there is a cutoff on the high-h_i values. For moderately relativistic systems, the important regions are near the (m_1, m_2) point.

We shall thus finally choose the 3D propagator

$$G^{\theta} \,=\, \hat{\theta}\, G^{SZ} \tag{67}$$

and replace V^{SZ} by $\hat{\theta}\, V^{\theta} \hat{\theta}$ in the various forms of Sazdjian's equation. In Breit's form, for example:

$$\psi^{\theta\overline{B}} \,=\, \frac{1}{P_0 - S + i\epsilon P_0}\; \frac{2\, P_0\, \hat{\theta}\, V^{\theta}\, \hat{\theta}}{2\,|P_0| + \hat{\theta}\, V^{\theta}\, \hat{\theta}}\; \psi^{\theta\overline{B}} \tag{68}$$

3.4 One-body limit.

If we go to the center of mass reference frame, write $P_0 = m_2 + W_1$ (for $P_0 > 0$) and make $m_2 \to \infty$ in (68), we get the equation

$$\psi^{\theta\overline{B}} \,=\, \frac{1}{W_1 - h_1 + i\epsilon}\; \theta(W_1 - |\,\vec{p}_1|)\; V^{\theta}\, \theta(W_1 - |\,\vec{p}_1|)\; \psi^{\theta\overline{B}} \tag{69}$$

as the limit of $\hat{\theta}$ is $\theta(W_1 - |\,\vec{p}_1|)$. Equation (69) is related to the equation

$$\psi \,=\, \frac{1}{W_1 - h_1 + i\epsilon}\, V\, \psi \tag{70}$$

which describes the light fermion 1 in the potential generated by the heavy fermion 2. We have

$$V^\theta \;=\; V\left(1 \;-\; \frac{\theta(|\,\vec{p}_1| - W_1)}{W_1 - h_1}\,V\right)^{-1} \tag{71}$$

$$\psi^\theta \;=\; \theta(W_1 - |\,\vec{p}_1|)\,\psi. \tag{72}$$

Equation (69) is thus the exact equation for the projection (72) provided the expansion of (71) is not truncated. If, however, the two-body potential is truncated to the Born term, its one-body limit will be V instead of V^θ. The introduction of the projector $\hat{\theta}$ has destroyed the exact one-body limit of the Born approximation, as it would also be the case with the more usual continuum dissolution preserving operators τ or Λ^{++}.

4 Two fermions in an external potential.

The two-fermion in an external potential problem is already much more complicated than the pure two-fermion problem, although it exhibits some simplifying features when compared with the three-fermion problem. The principal new difficulty is the non-conservation of the total spatial momentum $\vec{P}$. In the pure two-fermion case, this conservation law forbids the mixing of the physical bound states with the mixed-energy states (continuum dissolution). When an external potential is present, it becomes possible to get any given energy in an infinity of ways by combining fermions with opposite free energy signs. The positive-energy bound states will then not be normalizable, being mixed with a continuum.

The two-fermion plus potential equations can easily be obtained with the simple generalizations [23]

$$h_i \;=\; \vec{\alpha}_i\,.\vec{p}_i + \beta_i\,m_i \;+\; V_i(\vec{x}_i, \gamma_i) \tag{73}$$

where V_i is the external potential acting on fermion i (beyond the ladder approximation we should also modify the fermion propagators inside the crossed graphs). The continuum dissolution appears in the reduction using Breit's propagator: we get singularities in the higher-order terms of the 3D perturbation calculation and of the reduction series. The continuum dissolution is avoided with the reductions using $\tau, \Lambda^{++}, \sigma$ or $\hat{\theta}$ operators (which must also be built with the generalized h_i). Our projector $\hat{\theta}$ is still given by (65) in terms of the generalized h_i. Besides the free-free continuum, the (h_1, h_2) spectrum contains now bound-free or free-bound combinations (lines) and bound-bound combinations (points). The equations are written in the laboratory reference frame, defined as the reference frame in which the external potential is static. When the mutual interaction is switched off, we get a pair of independent Dirac equations in the external potential. When the external potential is switched off, we

get the equation of a system of mutually interacting fermions. This equation must no more refer to the laboratory reference frame, i.e. it must be invariant.

We have already studied the two-fermion in an external potential problem some years ago [23]. After the switching off of the external potential, the resulting equation was implicitly covariant provided the reduction series is not truncated: the 3D reduction being based on a non-covariant zero-order propagator, the truncations on the 3D potential led to non-covariant 3D equations. In the present work, with the combination of the covariant propagator of Sazdjian and our covariant $\hat{\theta}$ projector, the truncations of the reduction series are also covariant. We could thus keep only the Born term of V^θ in the various forms of the positive-energy Sazdjian equation. With the non-hermitian form, for exemple, we have

$$P_0\,\psi \;=\; \left[\, h_1 + h_2 + \;\frac{P_0 + h_1 + h_2}{2\,|\,P_0\,|}\;\hat{\theta}\;V^\theta\,\hat{\theta}\,\right]\,\psi. \tag{74}$$

If we start with a bound eigenstate of both h_i and consider the mutual interaction as a perturbation, we can, at first order, replace P_0 by h_1+h_2 before and after V^θ. The first-order energy shift will be

$$\Delta\,P_0 \;=< V^\theta > . \tag{75}$$

With the hermitian form we get the same result if we neglect $\hat{\theta}\,V^\theta\hat{\theta}$ before $2|P_0|$ in the denominator of (68). If we start with (58), we have

$$P_0\,\chi \;=\; \left(\, h_1 + h_2 + \sqrt{\frac{P_0+S}{2\,|P_0|}}\,\hat{\theta}\,V^\theta\,\hat{\theta}\,\sqrt{\frac{P_0+S}{2\,|P_0|}}\,\right)\,\chi. \tag{76}$$

In a perturbation calculation, the arguments of the first and last square roots of the terms of the perturbation series will be positive. If we go beyond the first order, the internal square roots will always appear in pairs.

5 Three fermions.

5.1 In search of a cluster separable 3D equation...

A three-fermion 3D equation, inspired by the two-fermion equation (29) with $A = 1$ could be

$$(P_0 - S)\,\psi \;=\; [\,V_{12}(P_0 - h_3) \;+\; V_{23}(P_0 - h_1) \;+\; V_{31}(P_0 - h_2)\,]\,\psi \tag{77}$$

with $S = h_1 + h_2 + h_3$. The potentials are the two-body potentials defined by (30). Each two-body potential V_{ij} was a function of the partial energy P_{ij0}, which we fixed to its (ij)+k cluster-separated limit $P_0 - h_k$ (in the following,

we shall omit the $(P_0 - h_k)$ arguments when not explicitly needed). At the $V_{23} = V_{31} = 0$ limit, for example, we get

$$\Big[P_0 - h_1 - h_2 - h_3 \Big] \, \psi \, = \, V_{12}(P_0 - h_3) \, \psi. \tag{78}$$

Writing

$$\psi = \psi_{12} \, \psi_3, \qquad P_0 \, = \, P_{120} \, + \, p_{30} \tag{79}$$

we get two independent equations:

$$\Big[P_{120} - h_1 - h_2 \Big] \, \psi_{12} \, = \, V_{12}(P_{120}) \, \psi_{12}, \qquad p_{30} \, \psi_3 \, = \, h_3 \, \psi_3 \tag{80}$$

as $P_0 - h_3$ can be replaced by $P_0 - p_{30} = P_{120}$. Our 3D equation (77) satisfies thus clearly the cluster separability requirement. Furthermore, the three cluster-separated limits are exactly the 3D equations we would get by 3D-reducing the corresponding two-fermion Bethe-Salpeter equations. This cluster separability is a property of the equation, or, if we prefer, of the full Green function. For a given scattering solution it is also possible to take the cluster-separated limit at fixed P. This is not possible for the bound state solutions.

In the two-body problem all quantities can be defined in the center of mass reference frame. In the two-body plus potential problem we had to start in the laboratory frame but to consider also the center of mass reference frame at the no-external potential limit. In the three-body problem we must start in the three-body center of mass frame and consider the center of mass reference frames of the three possible two-body subsystems obtained by cluster separation. The fact that the three cluster-separated limits are exact insures in principle the Lorentz invariance / cluster separability requirement: at the cluster separated limits the two-body equation can indeed be transformed back into a covariant two-body Bethe-Salpeter equation. There is no necessity of introducing Lorentz boosts by hand.

Equation (77) can however not be used, as it suffers of continuum dissolution. In [26], we made a careful examination of this problem. We found no mixing of the physical bound states with the two-cluster continua (a bound state of two fermions with a third fermion of opposite free energy sign), owing to the total spatial monentum conservation as in the pure two-fermion case. However, we found that the physical bound states $(+\,+\,+)$ can mix with the $(+\,+\,-)$, $(+\,-\,+)$ and $(-\,+\,+)$ states of the three-cluster continuum. The mixing can be avoided by writing a three-fermion 3D equation, inspired by the two-fermion equation (29) with $A = \Lambda^{++}$, such as

$$(P_0 - E_1 - E_2 - E_3) \, \psi \, = \, \Lambda^{+++} \, (\, V_{12}^{++} + V_{23}^{++} + V_{31}^{++} \,) \, \Lambda^{+++} \, \psi. \tag{81}$$

5.2 ...with covariant truncations.

Our two-body potentials are the sum of an infinity of contributions symbolized by Feynman graphs. Keeping only the first one (Born approximation) or a finite number of them would break the Lorentz covariance of the two-fermion clusters. A Born approximation preserving the Lorentz invariance / cluster separability property can be obtained by using the 3D reduction based on Sazdjian's covariant propagator combined with the covariant substitute $\hat{\theta}$ of Λ^{++} to prevent continuum dissolution. The result is

$$(P_0 - S)\,\psi \,=\, (\,U_{12} \,+\, U_{23} \,+\, U_{31}\,)\,\psi \tag{82}$$

with

$$U_{12} \,=\, \left[\frac{P_{120} \,+\, S_{120}}{2\,|P_{120}|}\;\hat{\theta}_{12}\,V_{12}^{\theta}\,\hat{\theta}_{12}\right]_{P_{120}=P_0-h_3}\,,\cdots \tag{83}$$

or with

$$U_{12} \,=\, \left[\frac{2\,P_{120}\,\hat{\theta}_{12}\,V_{12}^{\theta}\,\hat{\theta}_{12}}{2\,|P_{120}|\,+\,\hat{\theta}_{12}\,V_{12}^{\theta}\,\hat{\theta}_{12}}\right]_{P_{120}=P_0-h_3}\,,\cdots \tag{84}$$

if we start with the two-fermion positive-energy Sazdjian equation in Breit's hermitian form. Note that the resulting equations (82) obtained with the choices (83) or (84) are not equivalent, although they have equivalent two-cluster limits. Switching off the interactions between the third fermion and the cluster (12) will give a free Dirac equation for the third fermion, and one of the forms of the positive-energy Sazdjian equation written in the (123) center of mass reference frame for the (12) cluster. This last equation being covariant can be considered to be written in the (12) center of mass reference frame or in any other frame. This covariance is preserved even if we use a (Lorentz invariant) approximation of V_{12}^{θ}. We did not consider the form (58) of Sazdjian's equation, as here the arguments of the square roots could become negative.

6 Conclusions.

Using the explicitly covariant Sazdjian propagator combined with our covariant substitute of the positive-energy projector, we built a 3D equation for two fermions in an external potential and for three fermions. The two-fermion equations obtained at the cluster separated limits are these which would be obtained by a 3D reduction of the two-fermion Bethe-Salpeter equation. In our preceding works this implied that they were implicitly covariant if not truncated, even if they were written in a specific reference frame (the laboratory frame or the three-fermion rest frame). In the present work, however, they are explicitly covariant, which implies that the covariance will survive the possible truncations of the 3D potentials. The invariant substitute of the positive energy projector we use brings a cutoff on the spatial momenta, which could be welcome when

insuring the existence of some integrals. This cutoff appear for rather low, but nevertheless truly relativistic values. It is not by itself an approximation: it reflects the choice to work with a given projection of the two-fermion equation.

The 3D equations we write in this work are more complicated that the corresponding equations in our preceding works. Keeping in mind that in both cases the 3D potentials are given by series to be truncated anyway, we hope that the Lorentz invariance / cluster separability properties of the truncations of the present equations will result in a better approach of the real physics in the first terms.

References

[1] E.E. Salpeter and H.A. Bethe: Phys. Rev. **84** 1232 (1951)

[2] M. Gell-Mann and F. Low: Phys. Rev. **84** 350 (1951)

[3] E.E. Salpeter: Phys. Rev. **87** 328 (1952)

[4] A.A. Logunov and A.N. Tavkhelitze: Nuovo Cimento **29** 380 (1963)

[5] R. Blankenbecler and R. Sugar: Phys. Rev. **142** 1051 (1966)

[6] F. Gross: Phys. Rev. **186** 1448 (1969); Phys. Rev. **C26** 2203 (1982). F.Gross, J.W. Van Orden and K. Holinde: Phys. Rev. **C45** 2094 (1992)

[7] I.T. Todorov: Phys. Rev. **D3** 2351 (1971)

[8] C. Fronsdal and R.W. Huff: Phys. Rev. **D3** 933 (1971)

[9] A. Klein and T.S.H. Lee: Phys. Rev. **D10** 4308 (1974)

[10] G.P. Lepage: Phys. Rev. **A16** 863 (1977); G.P. Lepage: Ph D dissertation, SLAC Report n 212 (1978); W.E. Caswell and G.P. Lepage: Phys. Rev. **A18** 810 (1978)

[11] J.L. Gorelick and H. Grotch: Journ. Phys. G **3** 751 (1977)

[12] H. Sazdjian: J. Math. Phys. **28** 2618 (1987); **29** 1620 (1988)

[13] G.T. Bodwin, D.R. Yennie and M.A. Gregorio: Rev. Mod. Phys. **57** 723 (1985)
J.R. Sapirstein and D.R. Yennie, in *Quantum Electrodynamics* edited by T. Kinoshita (World Scientific, Singapore 1990) p.560.

[14] V.B. Mandelzweig and S.J. Wallace: Phys. Lett. **B197** 469 (1987)
S.J. Wallace and V.B. Mandelzweig: Nucl. Phys. **A503** 673 (1989)

[15] E.D. Cooper and B.K. Jennings: Nucl. Phys. **A483** 601 (1988)

[16] J.H. Connell: Phys. Rev. **D43** 1393 (1991)

[17] J. Bijtebier and J. Broekaert: Journal of Physics G **22** 559, 1727 (1996)

[18] J. Bijtebier and J. Broekaert: Nucl. Phys. **A612** 279 (1997)

[19] D.R. Phillips and S.J. Wallace: Phys. Rev. **C54** 507 (1996); Few-Body Systems **24** 175 (1998)

[20] G.E. Brown and D.G. Ravenhall: Proc. Roy. Soc. London **A208** 552 (1951)

[21] J. Sucher: Phys. Rev. **A22** 348 (1980); Phys. Rev. Lett. **55** 1033 (1985)

[22] M.H. Mittleman: Phys. Rev. **A39** 1 (1989)

[23] J. Bijtebier and J. Broekaert: Nuovo Cimento **A105** 351, 625 (1992)

[24] F. Gross: Phys. Rev. **C26** 2226 (1982)

[25] A. Stadler, F. Gross, M. Frank: Phys. Rev. **C56** 2396 (1997)

[26] J. Bijtebier: A three-fermion Salpeter equation. Preprint VUB/TENA/98-02 (hep-th/9809131) (1998). Submitted to Few-Body Systems

[27] J. Bijtebier: Higher-order corrections to the two and three-fermion Salpeter equations. Preprint VUB/TENA/9901 (hep-th/9903035). Submitted to Few-Body Systems.

[28] G. Breit: Phys. Rev. **34** 553 (1929), **36** 383 (1930), **39** 616 (1932)

[29] J. Mourad and H. Sazdjian: J. Phys. G **21** 267 (1995)